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Non-linear systems of multiple degrees of freedom under both additive and multiplicative random excitations

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Abstract

A quasi-linearization procedure is developed for non-linear systems under both additive and multiplicative white-noise excitations to obtain response statistical properties, including moments, correlation functions, and spectral densities. In the proposed procedure, only the system properties are linearized, while all excitations are kept unchanged. In particular, the basic characteristics associated with the multiplicative excitations are preserved, which is the key factor for the accuracy of the approach. Numerical examples are given, and the accuracy of the procedure is substantiated by comparing the analytical results with those obtained from Monte Carlo simulations.

1. Introduction

Many engineering systems are excited by dynamic loads, which can best be modelled as random processes, for example, a building under earthquake or wind forces, a ship subjected to sea wave forces, etc. The response of such a system, in terms of displacement, velocity, stress, or strain, is also a random process. To design or maintain such a system requires the knowledge of the system response in terms of statistical properties, including for example, mean values, mean-square values, correlation functions, and spectral densities. The mean and mean-square value of a random process reflect its average properties at one time instant, whereas the correlation function describes its average relationship at two time instants. The spectral density, which exists when the random process is weakly stationary, gives the energy distribution in the frequency domain. For non-linear systems under only additive random excitations, various solution techniques have been developed for obtaining the response statistical moments, among which the equivalent

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linearization method [1–3] is the most popular. Approximate methods have been developed [4–14] to obtain the spectral density of the response for cases of strong non-linearity, for which the linearization technique may not be applicable. The problem becomes more complicated when multiplicative excitations are present. Krenk et al. [15] applied the stochastic averaging method to obtain the response spectral density on the condition of a certain energy level, and then averaged it over the entire energy range. The method is applicable only to single-degree-of-freedom systems.

Realizing that the multiplicative excitation plays a crucial role in affecting the characteristics of the system response, its retention is important in an approximation procedure. Thus, a quasi-linearization procedure is proposed herein on this basis. In particular, non-linear damping and stiffness forces in the original system are replaced by the equivalent linear ones while excitation terms are kept unchanged. The replacement system is quasi-linear since multiplicative excitations are still present. The statistical moments can be solved exactly for the replacing quasi-linear system. For obtaining the correlation functions and the spectral densities, solution procedures are developed using the Ito differential rule [16] and an integral transformation. The quasi-linearization approach is applicable to multi-degree-of-freedom non-linear systems. Numerical examples show that the procedure yields quite accurate results in comparison with those obtained from Monte Carlo simulation, even for the case of strong non-linearity.

2. Analysis

Consider a stochastically excited system governed by the equations

$$\ddot{Y}_j + h_j(\mathbf{Y}, \dot{\mathbf{Y}}) = \sum_{i=1}^{n} [a_{ji}\dot{Y}_i\eta_i(t) + b_{ji}Y_i\gamma_i(t)] + \xi_j(t) \quad (j = 1, 2, \ldots, n), \tag{1}$$

where $h_j(\mathbf{Y}, \dot{\mathbf{Y}})$ include both damping and stiffness forces, and $\eta_i(t), \gamma_i(t)$ and $\xi_j(t)$ are Gaussian white noises. As indicated in Eq. (1), the multiplicative excitations appear in the linear terms. For simplicity, it is assumed that the additive excitations $\xi_j(t)$ are not correlated with the multiplicative excitations $\eta_i(t)$ and $\gamma_i(t)$. The first step in the quasi-linearization procedure is to replace the damping and stiffness forces in system (1) by linear forces, namely,

$$\ddot{Y}_j + \sum_{i=1}^{n} (\alpha_{ji}Y_i + \beta_{ji}\dot{Y}_i) = \sum_{i=1}^{n} [a_{ji}\dot{Y}_i\eta_i(t) + b_{ji}Y_i\gamma_i(t)] + \xi_j(t) \quad (j = 1, 2, \ldots, n). \tag{2}$$

The system described by Eq. (2) is said to be quasi-linear since the principle of superposition is not applicable due to the presence of the multiplicative excitations. Letting $Y_j = X_j$ and $\dot{Y}_j = X_{j+n}$, a set of Ito stochastic differential equations [17] is derived from Eq. (2) as follows:

$$\mathrm{d}X_j = X_{j+n}\,\mathrm{d}t, \quad \mathrm{d}X_{j+n} = \sum_{i=1}^{2n} C_{ji}X_i\,\mathrm{d}t + \sigma_j(\mathbf{X})\,\mathrm{d}B_j(t) \quad (j = 1, 2, \ldots, n), \tag{3}$$

where $B_j(t)$ are independent unit Wiener processes, and C_{ji} and $\sigma_j(\mathbf{X})$ can be derived from Eq. (2) by incorporating the Wong–Zakai correction terms, which are linear in the present case (see e.g., Ref. [3]).

The equivalent linear coefficients α_{ji} and β_{ji} in Eq. (2) are chosen to minimize the following mean-square differences:

$$E\left\{\left[h_j(\mathbf{Y},\dot{\mathbf{Y}})-\sum_{i=1}^{n}(\alpha_{ji}Y_i+\beta_{ji}\dot{Y}_i)\right]^2\right\}\quad (j=1,2,\ldots,n), \tag{4}$$

which leads to

$$E[\mathbf{X}\mathbf{X}']\begin{bmatrix}\boldsymbol{\alpha}'\\ \boldsymbol{\beta}'\end{bmatrix}=E[\mathbf{X}\mathbf{h}'], \tag{5}$$

where a prime denotes a matrix transposition, $\mathbf{X}=\{X_1X_2\ldots\ldots X_{2n}\}'$, $\boldsymbol{\alpha}=[\alpha_{ij}]$, $\boldsymbol{\beta}=[\beta_{ij}]$, and $\mathbf{h}=\{h_1h_2\ldots\ldots h_n\}'$. The terms in $E[\mathbf{X}\mathbf{X}']$ on the left-hand-side of Eq. (5) are the second order moments of the state variables. Assuming that functions $h_j(\mathbf{Y},\dot{\mathbf{Y}})$ are polynomials of Y_j and $\dot{Y}_j$, which is true for many practical problems, the terms in $E[\mathbf{X}\mathbf{h}']$ on the right-hand-side of Eq. (5) are moments of the state variables. If these moments can be calculated for the replacing system (2), then the unknown linear coefficients α_{ij} and β_{ij} can be solved from Eq. (5) by iteration.

Denote the kth order moments of the state variables by $m_{i_1i_2\ldots i_n}=E[X_1^{i_1}X_2^{i_2}\ldots X_n^{i_n}]$, where $i_1,i_2,\ldots,i_n$ are non-negative integers and $i_1+i_2+\cdots+i_n=k$. Using the Ito stochastic differential rule [16], a set of ordinary differential equations for the kth order moments $m_{i_1i_2\ldots i_n}$ can be obtained from Eq. (3). These equations are closed for a quasi-linear system; namely, they contain moments only up to the kth order. These moment equations can be solved exactly and sequentially from lower to higher orders. Linear ordinary differential equations with constant coefficients are required to be solved for moments in the transient non-stationary state. For stationary moments, only linear algebraic equations need to be solved. Since the statistical moments can be solved exactly for the replacement quasi-linear system (2), the equivalent linearization coefficients α_{ij} and β_{ij} can be calculated iteratively from the moment equations and Eq. (5). Therefore, an assumption for the response probability distribution is avoided.

It is known that, for a quasi-linear system, the stationary moments exist only under certain conditions and that more restrictive conditions are required for higher order moments. However, if the original non-linear system is stable in moments of a certain order, the equivalent quasi-linear system is expected to be also stable in moments of the same order. This is inferred purely from the physics point of view and no rigorous proof could be made.

To obtain the correlation functions for the quasi-linear system (2), multiply $X_k(t-\tau)$ on both sides of Eq. (3) and take the assembled average. We obtain a set of equations for the correlation functions $R_{ij}(\tau)=E[X_i(t)X_j(t-\tau)]$ as follows:

$$\frac{dR_{jk}(\tau)}{d\tau}=R_{j+n,k}(\tau),\quad \frac{dR_{j+n,k}(\tau)}{d\tau}=\sum_{i=1}^{2n}C_{ji}R_{ik}(\tau)$$
$$(j=1,2,\ldots,n;k=1,2,\ldots,2n). \tag{6}$$

Equation set (6) can be solved with initial conditions $R_{ij}(0)=E[X_iX_j]$, which have been obtained from the second order moment equations.

The spectral density $\Phi_{ij}(\omega)$ can be obtained as the Fourier transform of the correlation function $R_{ij}(\tau)$. However, it may be obtained without first solving $R_{ij}(\tau)$. A direct procedure to obtain the spectral densities is given below. Define the integral transformation

$$\bar{\Phi}_{ij}(\omega) = \Im[R_{ij}(\tau)] = \frac{1}{\pi}\int_0^{\infty} R_{ij}(\tau)e^{-i\omega\tau}\,d\tau. \tag{7}$$

It can be shown that

$$\Im\left[\frac{dR_{ij}(\tau)}{d\tau}\right] = i\omega\bar{\Phi}_{ij}(\omega) - \frac{1}{\pi}E[X_iX_j]. \tag{8}$$

Using Eqs. (7) and (8), Eq. (6) can be transformed to

$$\begin{aligned} &i\omega\bar{\Phi}_{jk}(\omega) - \frac{1}{\pi}E[X_jX_k] = \bar{\Phi}_{j+n,k}(\omega),\\ &i\omega\bar{\Phi}_{j+n,k}(\omega) - \frac{1}{\pi}E[X_{j+n}X_k] = \sum_{i=1}^{2n} C_{ji}\bar{\Phi}_{ik}(\omega) \quad (j = 1, 2, \ldots, n; k = 1, 2, \ldots, 2n). \end{aligned} \tag{9}$$

The set of $\bar{\Phi}_{ij}(\omega)$ can be solved from Eq. (9). The spectral density functions are then obtained from

$$\Phi_{ii}(\omega) = \mathrm{Re}[\bar{\Phi}_{ii}(\omega)], \quad \Phi_{ij}(\omega) = \tfrac{1}{2}[\bar{\Phi}_{ij}(\omega) + \bar{\Phi}_{ji}^{*}(\omega)], \tag{10}$$

where an asterisk denotes the complex conjugate. It is noticed that Eq. (9) is a set of complex linear algebraic equations, and analytical solutions can be obtained for low-dimensional systems. For high-dimensional cases, analytical solutions may be tedious, but numerical solutions can be carried out quite simply.

The correlation functions and spectral densities obtained for the replacing quasi-linear system (2) are approximations for the original non-linear system (1). The preservation of the multiplicative excitations in the quasi-linear system is a crucial feature of the present approximation procedure.

3. Examples

3.1. A single-degree-of-freedom system with non-linear damping and stiffness

Consider first a single-degree-of-freedom system with cubic non-linearity in both the damping and stiffness forces, namely,

$$\ddot{Y} + 2\alpha_1\dot{Y} + \lambda\dot{Y}^3 + \Omega_1^2 Y + \delta Y^3 = \dot{Y}\eta(t) + Y\gamma(t) + \xi(t). \tag{11}$$

The spectral densities of the Gaussian white noises $\eta(t), \gamma(t)$, and $\xi(t)$ are $K_{\eta\eta}$, $K_{\gamma\gamma}$, and $K_{\xi\xi}$, respectively, and it is assumed for convenience that the three excitations are uncorrelated with each other. The non-linear system (11) can be replaced by an equivalent quasi-linear system as follows:

$$\ddot{Y} + 2\alpha Y + \Omega^2 Y = \dot{Y}\eta(t) + Y\gamma(t) + \xi(t), \tag{12}$$

where α and Ω are two equivalent linearization coefficients. Letting $X_1 = Y$ and $X_2 = \dot{Y}$, a set of Ito stochastic equations are derived from Eq. (12) as follows:

$$\mathrm{d}X_1 = X_2\,\mathrm{d}t,$$

$$\mathrm{d}X_2 = (-2\alpha X_2 + \pi K_{\eta\eta} X_2 - \Omega^2 X_1)\,\mathrm{d}t + [2\pi(K_{\gamma\gamma} X_1^2 + K_{\eta\eta} X_2^2 + K_{\xi\xi})]^{1/2}\,\mathrm{d}B(t), \tag{13}$$

where $B(t)$ is a unit Wiener process. Using the Ito stochastic differential rule [16], equations for the second order moments are obtained as follows:

$$\begin{aligned}
\frac{\mathrm{d}m_{20}}{\mathrm{d}t} &= 2m_{11},\\
\frac{\mathrm{d}m_{11}}{\mathrm{d}t} &= m_{02} - \Omega^2 m_{20} - (2\alpha - \pi K_{\eta\eta})m_{11},\\
\frac{\mathrm{d}m_{02}}{\mathrm{d}t} &= -2\Omega^2 m_{11} - 2(2\alpha - \pi K_{\eta\eta})m_{02} + 2\pi(K_{\gamma\gamma} m_{20} + K_{\eta\eta} m_{02} + K_{\xi\xi}).
\end{aligned} \tag{14}$$

The stationary solutions of Eqs. (14) are

$$m_{20} = \frac{\pi K_{\xi\xi}}{2\Omega^2(\alpha - \pi K_{\eta\eta}) - \pi K_{\gamma\gamma}}, \quad m_{02} = \Omega^2 m_{20}, \quad m_{11} = 0. \tag{15}$$

The validity of Eqs. (15) requires

$$\alpha > \pi K_{\eta\eta}, \quad 2\Omega^2(\alpha - \pi K_{\eta\eta}) > \pi K_{\gamma\gamma}, \tag{16}$$

which are the stability conditions for the second order moments of the quasi-linear system (12). However, if both the non-linear coefficients λ and δ in the original system (11) are positive, system (11) is stable in moments of any orders. In this case, the quasi-linear system (12) is also stable and conditions in Eqs. (16) are expected to be satisfied.

By using the same procedure, some of the fourth order stationary moments are obtained as follows:

$$\begin{aligned}
m_{40} &= \frac{3\pi K_{\xi\xi} m_{20}}{\Delta_m}\left[2\Omega^2 + 3(\alpha - 2\pi K_{\eta\eta})(2\alpha - 3\pi K_{\eta\eta})\right],\\
m_{04} &= \frac{3\pi K_{\xi\xi}\Omega^2 m_{20}}{\Delta_m}\left\{2\Omega^4 + 3(2\alpha - 3\pi K_{\eta\eta})\left[\Omega^2(\alpha - \pi K_{\eta\eta}) - \pi K_{\gamma\gamma}\right]\right\},
\end{aligned} \tag{17}$$

where

$$\begin{aligned}
\Delta_m &= \Omega^4(4\alpha - 7\pi K_{\eta\eta}) + 6\Omega^2(\alpha - \pi K_{\eta\eta})(\alpha - 2\pi K_{\eta\eta})(2\alpha - 3\pi K_{\eta\eta})\\
&\quad - 3\pi K_{\gamma\gamma}\left[\Omega^2 + 3(\alpha - 2\pi K_{\eta\eta})(2\alpha - 3\pi K_{\eta\eta})\right].
\end{aligned} \tag{18}$$

Similar to the case of the second order moments, certain conditions must be satisfied for Eqs. (17) to be valid. These conditions will be met if λ and δ in the original non-linear system are positive.

The two linearization coefficients can be expressed according to Eq. (5) as

$$\alpha = \alpha_1 + \frac{\lambda m_{04}}{2m_{02}}, \quad \Omega^2 = \Omega_1^2 + \frac{\delta m_{40}}{m_{20}}. \tag{19}$$

Eqs. (15), (17) and (19) can be solved iteratively for α and Ω, as well as the second and fourth order moments.

The equations for the correlation functions $R_{11}(\tau)$ and $R_{21}(\tau)$ are obtained from Eqs. (6) as

$$\frac{\mathrm{d}R_{11}(\tau)}{\mathrm{d}\tau} = R_{21}(\tau), \quad \frac{\mathrm{d}R_{21}(\tau)}{\mathrm{d}\tau} = -\Omega^2 R_{11}(\tau) - (2\alpha - \pi K_{\eta\eta})R_{21}(\tau). \tag{20}$$

With the initial conditions $R_{11}(0) = m_{20}$ and $R_{21}(0) = m_{11} = 0$, we obtain

$$R_{11}(\tau) = m_{20} \exp\left[-(\alpha - \pi K_{\eta\eta}/2)\tau\right] \cos\sqrt{\Omega^2 - (\alpha - \pi K_{\eta\eta}/2)^2}\tau. \tag{21}$$

$R_{21}(\tau)$ is then obtained from the first equation of (20), and $R_{12}(\tau)$ and $R_{22}(\tau)$ can be determined similarly.

For the present example, Eqs. (9) are, specifically,

$$\begin{aligned}
&\mathrm{i}\omega\bar{\Phi}_{11}(\omega) - \bar{\Phi}_{21}(\omega) = \frac{1}{\pi}m_{20},\\
&\Omega^2\bar{\Phi}_{11}(\omega) + (2\alpha - \pi K_{\eta\eta} - \mathrm{i}\omega)\bar{\Phi}_{21}(\omega) = 0,\\
&\mathrm{i}\omega\bar{\Phi}_{12}(\omega) - \bar{\Phi}_{22}(\omega) = 0,\\
&\Omega^2\bar{\Phi}_{12}(\omega) + (\mathrm{i}\omega + 2\alpha - \pi K_{\eta\eta})\bar{\Phi}_{22}(\omega) = \frac{1}{\pi}m_{02}.
\end{aligned} \tag{22}$$

Spectral density functions can then be solved from Eqs. (22) as follows:

$$\begin{aligned}
&\Phi_{11}(\omega) = \frac{m_{11}}{\pi\Delta_s}\Omega^2(2\alpha - \pi K_{\eta\eta}), \quad \Phi_{22}(\omega) = \omega^2\Phi_{11},\\
&\Phi_{12}(\omega) = -\frac{m_{11}}{\pi\Delta_s}\Omega^2(\Omega^2 - \omega^2),
\end{aligned} \tag{23}$$

where

$$\Delta_s = (\Omega^2 - \omega^2)^2 + \omega^2(2\alpha - \pi K_{\eta\eta})^2. \tag{24}$$

The stationary moments of Eqs. (15) and (17), the correlation function of Eq. (21), and the spectral densities of Eqs. (23) are exact for the equivalent quasi-linear system (12); they are approximate solutions for the original non-linear system (11).

Numerical calculations are carried out for system (11) with $\alpha_1 = 0.4$, $\Omega_1 = 6$, and two different sets of non-linear coefficients λ and δ. One set of $\lambda = 0.1$ and $\delta = 5$ corresponds to a weak non-linearity, while another of $\lambda = 1$ and $\delta = 50$ represents a quite strong non-linearity. The spectral densities of the multiplicative excitations are $K_{\gamma\gamma} = 0.5$, $K_{\eta\eta} = 0.05$, respectively. Fig. 1 shows the stationary mean-square values of Y against the spectral density $K_{\xi\xi}$ of the additive excitation, calculated from the present quasi-linearization procedure. The spectral densities of Y for the system with an additive excitation level $K_{\xi\xi} = 1$ are depicted in Fig. 2 for the two different cases of non-linearity. Also depicted in the figures are results obtained from Monte Carlo simulation. It is seen that the results calculated by using the proposed procedure agree quite well with those obtained from Monte Carlo simulation.

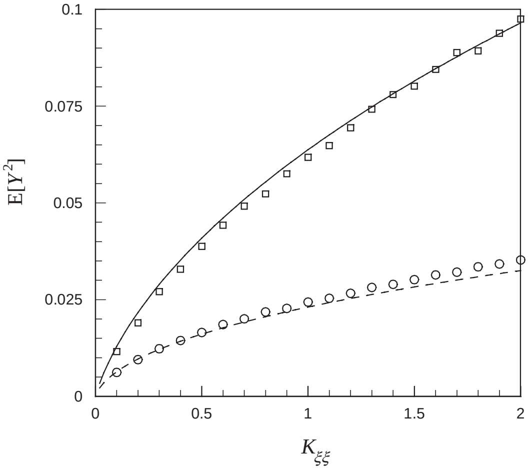

Fig. 1. Stationary mean-square values of response Y for system (11): —, analytical results, $\lambda = 0.1$, $\delta = 5$; □, simulation results, $\lambda = 0.1$, $\delta = 5$; - - - -, analytical results, $\lambda = 1$, $\delta = 50$; ∘, simulation results, $\lambda = 1$, $\delta = 50$.

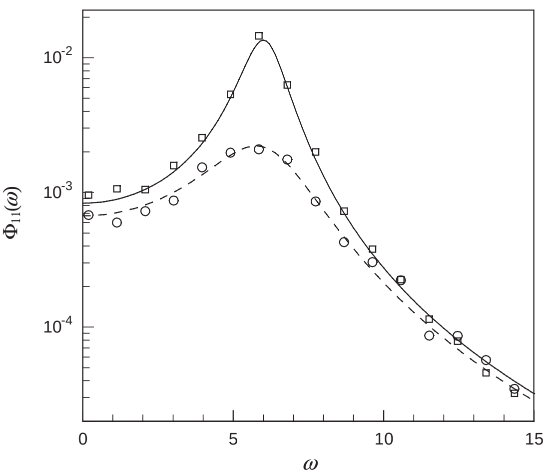

Fig. 2. Spectral densities of response Y for system (11) with $K_{\xi\xi} = 1$: —, analytical results, $\lambda = 0.1$, $\delta = 5$; □, simulation results, $\lambda = 0.1$, $\delta = 5$; - - - -, analytical results, $\lambda = 1$, $\delta = 50$; ∘, simulation results, $\lambda = 1$, $\delta = 50$.

3.2. A two-degree-of-freedom non-linear system with coupled multiplicative excitations

The second example is a two-degree-of-freedom system:

$$\begin{aligned} &\ddot{Y}_1 + 2\alpha \dot{Y}_1 + \lambda \dot{Y}_1^3 + \omega_1^2 Y_1 = \omega_1 \omega_2 Y_2 \eta(t) + \xi(t), \\ &\ddot{Y}_2 + 2\varsigma_2 \omega_2 \dot{Y}_2 + \omega_2^2 Y_2 = \omega_1 \omega_2 Y_1 \eta(t), \end{aligned} \tag{25}$$

where $\eta(t)$ and $\xi(t)$ are independent Gaussian white noises. These equations describe the fundamental modes of the transverse deflection and the angle of twist for a simply supported beam of narrow rectangular cross-section subjected to randomly varying transverse force and end moments, and undergoing bending and torsion [18,3]. The damping force for the bending motion Y_1 is assumed to be non-linear. Applying the quasi-linearization procedure, Eqs. (25) are replaced by

$$\begin{aligned}\ddot{Y}_1 + 2\varsigma_1\omega_1\dot{Y}_1 + \omega_1^2 Y_1 &= \omega_1\omega_2 Y_2\eta(t) + \xi(t),\\ \ddot{Y}_2 + 2\varsigma_2\omega_2\dot{Y}_2 + \omega_2^2 Y_2 &= \omega_1\omega_2 Y_1\eta(t).\end{aligned} \tag{26}$$

Letting $X_1 = Y_1, X_2 = Y_2, X_3 = \dot{Y}_1, X_4 = \dot{Y}_2$, and $m_{ijkl} = \mathrm{E}[X_1^i X_2^j X_3^k X_4^l]$, we obtain the second order moments for system (25) as

$$\begin{aligned}m_{2000} &= \frac{2\pi\varsigma_2 K_{\xi\xi}}{\omega_1^3(4\varsigma_1\varsigma_2 - \pi^2\omega_1\omega_2 K_{\eta\eta}^2)}, \quad m_{0020} = \omega_1^2 m_{2000},\\ m_{0200} &= \frac{\pi\omega_1^2 K_{\eta\eta}}{2\varsigma_2\omega_2} m_{2000}, \quad m_{0002} = \omega_2^2 m_{0200}.\end{aligned} \tag{27}$$

The other second moments are zero. The validity of Eqs. (27) requires

$$4\varsigma_1\varsigma_2 > \pi^2\omega_1\omega_2 K_{\eta\eta}^2, \tag{28}$$

which is assumed to be satisfied.

The analytical expressions for the fourth order moments can also be obtained exactly, but rather tediously. However, their numerical solutions are quite simple if it is assumed that the stability conditions for the fourth order moments are met. The damping coefficient in the linearized equation for the bending mode Y_1 is then

$$\varsigma_1 = \frac{\alpha}{\omega_1} + \frac{\lambda m_{0040}}{2\omega_1 m_{0020}} \tag{29}$$

which can be determined by iteration. The spectral densities of Y_1 and Y_2 are obtained as

$$\Phi_{11}(\omega) = \frac{2\varsigma_1\omega_1^3 m_{2000}}{\pi[(\omega_1^2 - \omega^2)^2 + 4\varsigma_1^2\omega_1^2\omega^2]}, \quad \Phi_{22}(\omega) = \frac{2\varsigma_2\omega_2^3 m_{0200}}{\pi[(\omega_2^2 - \omega^2)^2 + 4\varsigma_2^2\omega_2^2\omega^2]}. \tag{30}$$

It is noted that each spectral density in Eqs. (30) has the same form of a single-degree-of-freedom linear system. The coupling effects between the two modes are accounted for in the linearization coefficient ς_1 and the second order moments m_{2000} and m_{0200}.

Fig. 3 shows the stationary mean-square values of the bending mode Y_1 with a varying $K_{\xi\xi}$ for system (25) with the parameters $\omega_1 = 6$, $\omega_2 = 20$, $\alpha = 0.6$, $\zeta_2 = 0.1$, $K_{\eta\eta} = 0.004$, and two different values of non-linear coefficient $\lambda = 0.1$ and 1. Fig. 4 depicts the spectral densities of Y_1 calculated for the same system with $K_{\xi\xi} = 1$. Results obtained from Monte Carlo simulation are shown also in these figures to substantiate the accuracy of the analytical solutions.

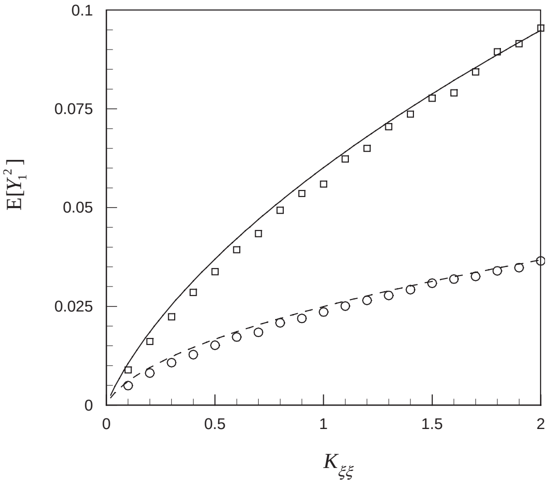

Fig. 3. Stationary mean-square values of response Y_1 for system (25): —, analytical results, $\lambda = 0.1$; □, simulation results, $\lambda = 0.1$; - - - -, analytical results, $\lambda = 1$; ∘, simulation results, $\lambda = 1$.

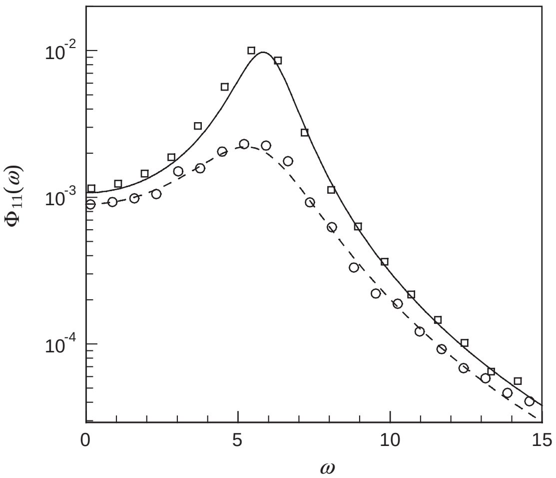

Fig. 4. Spectral densities of response Y_1 for system (25) with $K_{\xi\xi} = 1$: —, analytical results, $\lambda = 0.1$; □, simulation results, $\lambda = 0.1$; - - - -, analytical results, $\lambda = 1$; ∘, simulation results, $\lambda = 1$.

3.3. A two-degree-of-freedom non-linear system with coupled damping and stiffness

The third example is also a two-degree-of-freedom system with coupled damping and stiffness governed by

$$\begin{aligned} & m_1\ddot{Y}_1 + c_1\dot{Y}_1 + k_1 Y_1 + c_2(\dot{Y}_1 - \dot{Y}_2) + k_2(Y_1 - Y_2) + \beta_1(Y_1 - Y_2)^3 = Y_1 W_1(t) + W_2(t), \\ & m_2\ddot{Y}_2 + c_2(\dot{Y}_2 - \dot{Y}_1) + k_2(Y_2 - Y_1) + \beta_1(Y_2 - Y_1)^3 = 0, \end{aligned} \tag{31}$$

where $W_1(t)$ and $W_2(t)$ are two independent Gaussian white noises. Eqs. (31) can be used to describe a primary–secondary system with linear coupling in damping and non-linear coupling in stiffness, and the primary system is subjected to both additive and multiplicative excitations. By denoting

$$\rho=\frac{m_2}{m_1},\quad \omega_1^2=\frac{k_1}{m_1},\quad \varsigma_1=\frac{c_1}{2m_1\omega_1},\quad \omega_2^2=\frac{k_2}{m_2},\quad \varsigma_2=\frac{c_2}{2m_2\omega_2},$$
$$\beta=\frac{\beta_1}{m_2},\quad \eta(t)=\frac{W_1(t)}{m_1},\quad \xi(t)=\frac{W_2(t)}{m_1}, \tag{32}$$

Eqs. (31) are transformed to

$$\ddot{Y}_1+2\varsigma_1\omega_1\dot{Y}_1+\omega_1^2Y_1+2\rho\varsigma_2\omega_2(\dot{Y}_1-\dot{Y}_2)+\rho\omega_2^2(Y_1-Y_2)+\rho\beta(Y_1-Y_2)^3=Y_1\eta(t)+\xi(t),$$
$$\ddot{Y}_2+2\varsigma_2\omega_2(\dot{Y}_2-\dot{Y}_1)+\omega_2^2(Y_2-Y_1)+\beta(Y_2-Y_1)^3=0. \tag{33}$$

It is clear in Eqs. (32) that ρ is the mass ratio of the secondary and primary systems. Letting $X_1=Y_1, X_2=Y_2, X_3=\dot{Y}_1, X_4=\dot{Y}_2$, the Ito equations for the equivalent quasi-linear are

$$\mathrm{d}X_1=X_3\,\mathrm{d}t,$$
$$\mathrm{d}X_3=(-c_{11}X_3-c_{12}X_4-k_{11}X_1-k_{12}X_2)\,\mathrm{d}t+\sqrt{2\pi K_{\eta\eta}X_1}\,\mathrm{d}B_\eta(t)+\sqrt{2\pi K_{\xi\xi}}\,\mathrm{d}B_\xi(t),$$
$$\mathrm{d}X_2=X_4\,\mathrm{d}t,$$
$$\mathrm{d}X_4=(-c_{21}X_3-c_{22}X_4-k_{21}X_1-k_{22}X_2)\,\mathrm{d}t, \tag{34}$$

where $K_{\eta\eta}$ and $K_{\xi\xi}$ are spectral densities of $\eta(t)$ and $\xi(t)$, respectively, $B_\eta(t)$ and $B_\xi(t)$ are two independent unit Wiener processes, $c_{11}=2\varsigma_1\omega_1+2\rho\varsigma_2\omega_2$, $c_{12}=-2\rho\varsigma_2\omega_2$, $c_{21}=-2\varsigma_2\omega_2$, $c_{22}=2\varsigma_2\omega_2$, and

$$k_{11}=\omega_1^2+\rho\omega_2^2+\rho\beta\frac{m_{4000}-3m_{3100}+3m_{2200}-m_{1300}}{m_{2000}},$$
$$k_{12}=-\rho\omega_2^2+\rho\beta\frac{m_{3100}-3m_{2200}+3m_{1300}-m_{0400}}{m_{0200}},$$
$$k_{21}=-\omega_2^2+\beta\frac{m_{1300}-3m_{2200}+3m_{3100}-m_{4000}}{m_{2000}},$$
$$k_{22}=\omega_2^2+\beta\frac{m_{4000}-3m_{1300}+3m_{2200}-m_{3100}}{m_{0200}}. \tag{35}$$

Thus, the linearization coefficients k_{ij} in Eqs. (35), the stationary moments, the correlation functions, as well as the spectral densities can be solved using the proposed procedure.

Numerical calculations were carried out for system (31) with the parameters $\omega_1=3$, $\omega_2=6$, $\varsigma_1=\varsigma_2=0.05$, $\beta=0.5$. Fig. 5 shows the mean-square values of Y_1 with respect to a varying spectral density $K_{\eta\eta}$ of the multiplicative excitation. The mass ratio $\rho=0.1$, and the spectral density $K_{\xi\xi}$ of the additive excitation takes two different values of 0.5 and 1, respectively. It is seen that the spectral densities of both the additive and multiplicative excitations play important roles. The mean-square values of Y_1 are also depicted in Fig. 6 for $K_{\xi\xi}=1$ and two different mass ratios $\rho=0.1$ and 0.5, respectively. Fig. 6 shows that the mass ratio has a moderate influence on the mean-square value of the primary system motion Y_1. Fig. 7 shows the spectral density functions of the response Y_1 for cases of $K_{\eta\eta}=0.2$, $K_{\xi\xi}=1$, and two different mass ratios $\rho=0.05$ and 0.5,

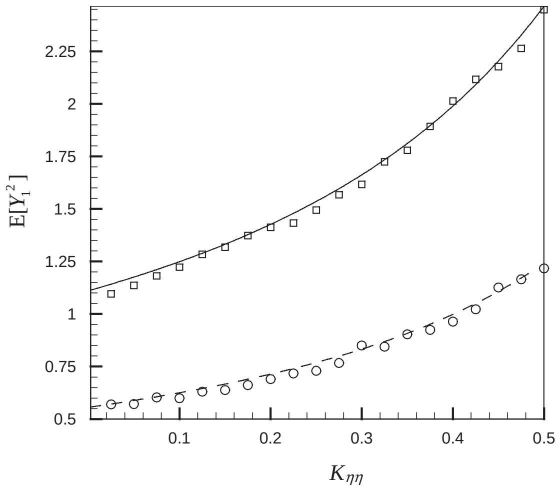

Fig. 5. Stationary mean-square values of response Y_1 for system (31) with $\rho = 0.1$: —, analytical results, $K_{\xi\xi} = 1$; □, simulation results, $K_{\xi\xi} = 1$; - - - -, analytical results, $K_{\xi\xi} = 0.5$; ∘, simulation results, $K_{\xi\xi} = 0.5$.

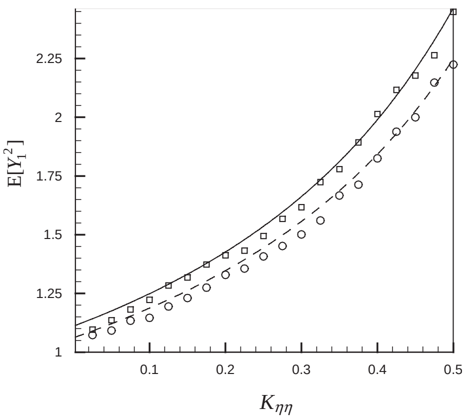

Fig. 6. Stationary mean-square values of response Y_1 for system (31) with $K_{\xi\xi} = 1$: —, analytical results, $\rho = 0.1$; □, simulation results, $\rho = 0.1$; - - - -, analytical results, $\rho = 0.5$; ∘, simulation results, $\rho = 0.5$.

respectively. The larger the mass ratio ρ is, the more significant effect the secondary system has on the frequency distribution of the primary system motion Y_1. Figs. 5–7 indicate a rather high accuracy of the proposed quasi-linearization procedure when comparing the analytical results with those from Monte Carlo simulations.

4. Concluding remarks

It is shown that the statistical moments, correlation functions, and spectral densities of responses can be obtained exactly for a quasi-linear system under Gaussian white-noise

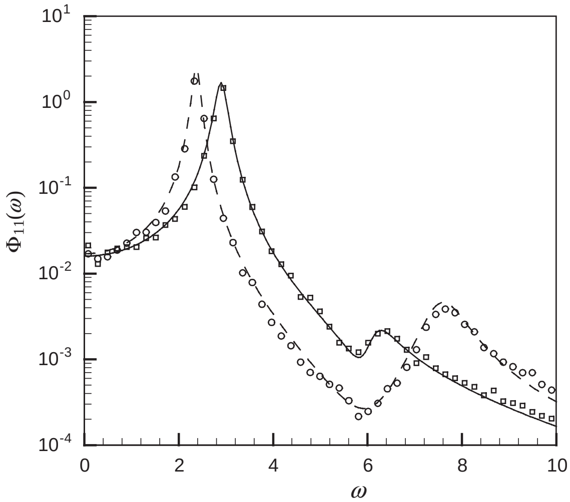

Fig. 7. Spectral densities of response Y_1 for system (31) with $K_{\eta\eta} = 0.2$ and $K_{\xi\xi} = 1$: —, analytical results, $\rho = 0.05$; □, simulation results, $\rho = 0.05$; - - - -, analytical results, $\rho = 0.5$; ◦, simulation results, $\rho = 0.5$.

excitations. Taking advantage of this result, a quasi-linearization procedure is proposed to replace a non-linear system excited by both additive and multiplicative excitations by an equivalent quasi-linear one, and then solve it for the approximate statistical properties of the original non-linear system. Since only linear algebraic equations need to be solved for both the stationary moments and the spectral density functions, the approach is feasible for multi-degree-of-freedom systems. One important feature of the procedure is the preservation of the multiplicative excitations in the linearized equations, which accounts for its accuracy as compared with results from Monte Carlo simulation.

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